

# Emergence of complex and spinor wave functions in scale relativity. I. Nature of scale variables

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## Abstract

One of the main results of Scale Relativity as regards the foundation of quantum mechanics is its explanation of the origin of the complex nature of the wave function. The Scale Relativity theory introduces an explicit dependence of physical quantities on scale variables, founding itself on the theorem according to which a continuous and non-differentiable space-time is fractal (i.e., scale-divergent). In the present paper, the nature of the scale variables and their relations to resolutions and differential elements are specified in the non-relativistic case (fractal space). We show that, owing to the scale-dependence which it induces, non-differentiability involves a fundamental two-valuedness of the mean derivatives. Since, in the scale relativity framework, the wave function is a manifestation of the velocity field of fractal space-time geodesics, the two-valuedness of velocities leads to write them in terms of complex numbers, and yields therefore the complex nature of the wave function.

## 1 Introduction

From a physical point of view, the Scale Relativity theory is the generalization to scales of the relativity principle of motion which underlies the foundation of a large part of classical physics. From a mathematical point of view, it is the giving up of the hypothesis of space-(time) differentiability. Both generalizations result in the fractal nature of space-(time).

One of this theory main achievements is the foundation of quantum mechanics on first principles. In its framework, the quantum mechanical postulates have been derived and the complex, then spinorial, then bi-spinorial nature of the wave function has been naturally recovered [1, 2, 3, 4, 5], while the corresponding quantum mechanical motion equations, the Schrödinger [1], Pauli [3], Klein-Gordon [6, 7] and Dirac [4, 5] equations have been demonstrated.

The theory has also allowed us to generalize the quantum mechanical motion equations to the macroscopic realm. This is obtained when the constant  $\mathcal{D}$  which characterizes the amplitude of the fractal fluctuations appearing in the theory, and which corresponds to  $\mathcal{D} = \hbar/2m$  in the standard quantum

theory, is given a more general interpretation in terms of a macroscopic constant whose value is linked to the physical system under study. We have been therefore able to derive a macroscopic Schrödinger-like equation with numerous applications in physics and other fields (see [8] and references therein) and a macroscopic Dirac-like equation whose non-relativistic limit might tentatively reproduce some turbulent fluid behavior [9].

The emergence of the complex numbers and their generalizations, the quaternions and the bi-quaternions, is issued from the successive doublings of the velocity fields due to the non-differentiability of the fractal functions representing the space-(time) coordinates. These doublings have been extensively studied in previous works (see, e.g., [8] for a recent review).

We wish to give here however a new and more detailed derivation of the way the two-valuedness of the mean velocity field naturally emerges in terms of a (+) and (-) velocity, yielding  $V$  and  $U$  velocity fields, which are subsequently combined in terms of the complex velocity field  $\mathcal{V} = V - iU$  used in the geodesic equation  $d\mathcal{V}/dt = 0$  from which the quantum mechanical motion equations are derived. Then, we are led to generalize these results to three dimensions, being therefore allowed to recover the complex 3-velocity field needed to obtain the Schrödinger equation [1].

In a subsequent paper [10], we will be led to extend these results to the 4-velocity field needed to obtain the Klein-Gordon equation [6, 7]. Then we will apply this two-valuedness to successive other doublings, obtaining thus naturally a new expression for the bi-quaternionic velocity field yielding the bi-spinor and the Dirac equation [4, 5], from which the spinor and the Pauli equation are derived [3].

The structure of the present paper is as follows. In Sec. 2, we give a physically complete picture of how the complex numbers emerge in the quantum mechanical domain from the doubling of the velocity fields in a non-differentiable space. In Sec. 3, we generalize to three dimensions the one-dimensional results previously obtained in Sec. 2. In Sec. 4, we present our conclusions.

## 2 Emergence of complex numbers in one dimension

### 2.1 General argument

The first step is the theorem according to which a continuous and non-differentiable curve is fractal, in the general meaning that its length is explicitly scale dependent and tends to infinity when the resolution interval tends to 0. In terms of a parameter  $t$  along this curve (which may, in particular, be a time coordinate), and taking as resolution variable small intervals  $dt$  of the parameter  $t$ , it reads ( $L = L(t, dt) \rightarrow \infty$ ) $_{dt \rightarrow 0}$ .

A continuous fractal coordinate therefore reads  $X = X(t, dt)$  and it is non-differentiable, under the meaning that, though differential elements  $dX$  and  $dt$  can be defined as a consequence of continuity, their ratio  $dX/dt \rightarrow \infty$  when  $dt \rightarrow 0$ . We have therefore suggested [1] to redefine the derivative as a fractal function, i.e., as a function which is explicitly dependent on the scale interval  $dt$ . But this new definition is now two-valued:

$$V_+(t, dt) = \frac{d_+ X}{dt} = \left( \frac{X(t + dt, dt) - X(t, dt)}{dt} \right)_{dt \rightarrow 0_+}, \quad (1)$$

$$V_-(t, dt) = \frac{d_- X}{dt} = \left( \frac{X(t + dt, dt) - X(t, dt)}{dt} \right)_{dt \rightarrow 0_-}. \quad (2)$$

This generalized definition of the derivative includes the standard differential one, in which one takes the limit  $dt \rightarrow 0$ , as a particular case. Indeed, if the limit exists, as in the differentiable case, then it is included in this definition as  $V(t) = V_+(t, 0) = V_-(t, 0)$ . If it does not exist, the standard definition fails, since the derivative is in this case undefined, while the new definition, which includes all the history of the way the function behaves when  $dt \rightarrow 0$  (including its divergence) is effective as a tool with which one can work.

In terms of the above definition, in which  $dt$  is allowed to be positive or negative, there is, strictly, only one function,  $V(t, dt)$ . The two-valuedness is a manifestation of the fact that, in general, there is no reason why it should be symmetric with respect to the variable  $dt$ . However, from the view point of scale laws, i.e., when looking to a specific transformation  $dt \rightarrow dt'$ , the natural scale variable is not  $dt$  but its logarithm, which acts as a kind of theoretical magnifying glass to view the behavior of the function around  $dt = 0$ . This implies to jump to its absolute value  $|dt|$  and to choose a reference scale  $\tau$  in order to write it as  $\rho_t = \ln(|dt|/\tau)$ . This writing is just a manifestation of the principle of scale relativity, according to which scales do not exist as such, but only through their ratios. If we want now to plot the function  $V(t, \rho_t)$ , it is clear that we are obliged to use two functions,  $V_+ = [V(t, \rho_t)]_{dt>0}$  and  $V_- = [V(t, \rho_t)]_{dt<0}$ .

Another result of the scale relativity approach is that such a scale-dependent fractal function can be generally written as the sum of a differentiable “classical” part and of a non-differentiable, divergent fractal part. In the case of a fractal velocity field, this reads

$$V_+[x(t, dt), t, dt] = v_+[x(t), t] + w_+[x(t, dt), t, dt], \quad (3)$$

$$V_-[x(t, dt), t, dt] = v_-[x(t), t] + w_-[x(t, dt), t, dt]. \quad (4)$$

where  $w_{\pm} = d\xi_{\pm}/dt$  and  $|d\xi_{\pm}|/|dt|^{D_f} \propto |dt|$ ,  $D_f$  being the fractal dimension of the geodesics.

In previous publications, we have considered that, in general, there is no a priori reason for the “classical” velocity fields  $v_+$  and  $v_-$  to be the same. It is this specific point that we want to address and to elaborate here, in order to understand better the reason why this two-valuedness affects not only the fractal (scale-dependent) part of the velocity, but also its “classical”, scale-independent, differentiable part.

## 2.2 Two-valuedness of the mean velocity field

Let us consider a fractal coordinate  $X$ . By “fractal”, we mean that it is explicitly dependent on a scale variable (or “resolution”)  $\varepsilon > 0$  and divergent when  $\varepsilon \rightarrow 0$ . Being defined as a resolution,  $\varepsilon$  is fundamentally positive, i.e.,  $\varepsilon \in \mathbb{R}^+$ . This resolution may be of two types, time-like ( $\varepsilon_t$ ) or space-like ( $\varepsilon_x$ ). The relation between the time-resolution and space-resolution on a fractal curve of fractal dimension  $D_f$  is:

$$\varepsilon_x^{D_f} \propto \varepsilon_t. \quad (5)$$

Consider the case when the curve described by the curvilinear coordinate  $X$  is traveled during time  $t$ . In terms of the time-resolution, the coordinate then reads  $X = X(t, \varepsilon_t)$ . For any given value of  $\varepsilon_t$ , the curve is smooth and differentiable.

Then it is quite possible to define a time differential element  $dt \rightarrow 0$  on this curve. The time measurement resolution  $\varepsilon_t$  defines the transition between what is considered as the time variable  $t > \varepsilon_t$ , which is measured in units of this resolution and the differential element  $dt \leq \varepsilon_t$  tending to 0 (see Fig. 1).

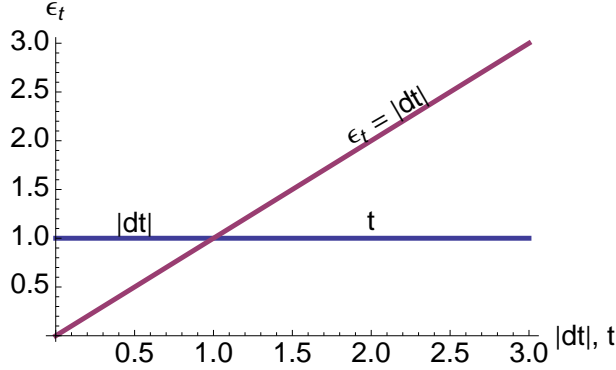


Figure 1: Time measurements on a physical "object" and their theoretical description involve the measurement resolution  $\varepsilon_t$ , the time variable  $t$  and the time differential element  $dt$ . The same is true for space  $x$  and other measurements. Values of measurement results are by construction  $> \varepsilon_t$  and are represented by a time variable  $t$ . Differential elements used in a theoretical description are therefore such that  $dt \leq \varepsilon_t$ . When the object is explicitly resolution-dependent ("fractal" in a general meaning), it changes with the value of  $\varepsilon_t$ , so that the differential element  $dt$  is no longer defined on the same object. A solution to this problem consists in placing oneself on the resolution "interface", e.g., to work with differential elements equal to the resolution, i.e., such that  $|dt| = \varepsilon_t$ .

For this curve, observed, measured or considered at a fixed resolution  $\varepsilon_t$ , we may then define a mathematical velocity in the standard differentiable way:

$$\tilde{V}_{\varepsilon_t}(t) = \frac{dX_{\varepsilon_t}}{dt} = \lim_{dt \rightarrow 0} \left( \frac{X_{\varepsilon_t}(t + dt) - X_{\varepsilon_t}(t)}{dt} \right). \quad (6)$$

We have written here  $\varepsilon_t$  as an index, since it is fixed at a given value, instead of being a variable. It must be understood that this derivative is a purely mathematical tool, valid only on the curve of resolution  $\varepsilon_t$  and allowing to make differential calculus on this curve, but that it does not represent the true velocity on the fractal curve. Indeed, when the time resolution interval is decreased ( $\varepsilon'_t < \varepsilon_t$ ), completely new information appears on the fractal curve, which was unpredictable from the sole knowledge of  $X_{\varepsilon_t}$ . A new velocity  $\tilde{V}_{\varepsilon'_t}(t)$  can now be defined, which may be fundamentally different from  $\tilde{V}_{\varepsilon_t}(t)$ . This derivative has only mathematical meaning at scales smaller than  $\varepsilon'_t$ , and physical meaning only at scale  $\varepsilon'_t$ .

Considering again  $\varepsilon_t$  as an explicit variable, this means that one can define a "fractal velocity" as

$$V_f(t, dt, \varepsilon_t) = \frac{dX}{dt} = \frac{X(t + dt, \varepsilon_t) - X(t, \varepsilon_t)}{dt}, \quad (7)$$

where  $\varepsilon_t > 0$ , while  $dt$  is algebraic and can be positive or negative, i.e.,  $dt \in \mathbb{R}$ .

It is now clear from the above discussion that there is a particularly meaningful choice for the scale variable, which is  $\varepsilon_t = |dt|$ . This special choice amounts to place oneself just on the interface where the derivative takes its physical value whatever the scale. This choice corresponds to the natural definition of a resolution from the theoretical viewpoint, in accordance with the Riemann-Lebesgue method (while from the experimental viewpoint, resolutions can be intervals like in a Chrapak multi-wire detector, but may also be defined as pixels, or covering balls, standard errors in statistical measurements, etc., see [8, Chap. 3]). The

interval  $dt$  has therefore two different and complementary roles: (1) as a differential element, it appears in sums like  $t + dt$  and is algebraic; (2) as a resolution interval, its sign loses its meaning and it appears in scale transformations in a logarithmic form, such as  $\ln(|dt|/\tau)$ .

More generally, in a fractal space, we are led to consider a fractal velocity field  $V(x(t), t, dt, \varepsilon_t) = dX/dt$ , so that the differential element  $dX(x(t), t; dt, \varepsilon_t)$  can be decomposed in terms of various contributions, a differentiable linear one and a non-linear fractal fluctuation:

$$dX = V[x(t), t] dt + U[x(t), t] \varepsilon_t + d\xi \quad (8)$$

The fractal fluctuation can be described by a stochastic variable which has the nature of a space resolution  $\varepsilon_x > 0$ , i.e.,

$$d\xi = \eta \varepsilon_x, \quad (9)$$

where  $\eta$  is a purely mathematical dimensionless variable which is normalized according to  $\langle \eta \rangle = 0$  and  $\langle \eta^2 \rangle = 1$  and may have any probability distribution. When the fractal dimension is  $D_f = 2$ , the space and time resolutions are related by

$$\varepsilon_x = \sqrt{2\mathcal{D}\varepsilon_t}. \quad (10)$$

We obtain therefore

$$dX = V[x(t), t] dt + U[x(t), t] \varepsilon_t + \eta \sqrt{2\mathcal{D}\varepsilon_t}. \quad (11)$$

If we place ourselves now on the interface  $\varepsilon_t = |dt|$  which defines the physical fractal derivative, we have

$$dX = V dt + U |dt| + \eta \sqrt{2\mathcal{D}|dt|}. \quad (12)$$

Therefore two possibilities occur for the elementary displacements:

$$dt > 0 : \quad |dt| = dt, \quad d_+ X = (V + U) dt + \eta \sqrt{2\mathcal{D}dt}. \quad (13)$$

$$dt < 0 : \quad |dt| = -dt, \quad d_- X = (V - U) dt + \eta \sqrt{-2\mathcal{D}dt}. \quad (14)$$

By setting  $v_+ = V + U$  and  $v_- = V - U$ , we recover the two-valuedness of the mean velocity field in terms of a (+) and (-) velocity. But in this new and more detailed derivation, it is noticeable that the  $V$  and  $U$  velocity fields, which are subsequently combined in terms of the complex velocity field  $\mathcal{V} = V - iU$  used in the geodesic equation  $d\mathcal{V}/dt = 0$ , appear first.

They acquire here a new status,  $V$  being the component linked to the algebraic differential element  $dt$  while  $U$  is the component linked to  $|dt|$ , now understood as a scale variable having the properties of a resolution interval submitted to scale transformations  $\ln(|dt|/\tau) \rightarrow \ln(|dt'|/\tau)$ . This result reinforces the choice of identifying  $V = (v_+ + v_-)/2$  with the real part of the complex velocity. Indeed, in the new writing,  $dX = V dt$  is the term obtained in standard differentiable calculus, while the additional term  $U \varepsilon_t$  is new and linked to the explicit scale dependence. Moreover,  $V[x(t), t]$  is the velocity field that naturally appears in the fluid mechanics form of the Schrödinger equation (continuity equation + Euler equation including a quantum potential [11]).

### 3 Generalization to three dimensions of the one-dimensional results

To derive the Schrödinger equation, a generalization of the one-dimensional results obtained above to the three dimensional space is needed. For each of the

three coordinates defining this space, we can write Eq. (8) as

$$dX_i = V_i[x_i(t), t]dt + U_i[x_i(t), t]\varepsilon_t + d\xi_i, \quad (15)$$

with  $i = 1, 2, 3$ ,  $d\xi_i = \eta_i \varepsilon_{x_i}$ ,  $\varepsilon_{x_i} = \sqrt{2\mathcal{D}\varepsilon_t}$ , and the dimensionless variables  $\eta_i$  being normalized according to  $\langle \eta_i \rangle = 0$ ,  $\langle \eta_i \eta_j \rangle = \delta_{ij}$ , where  $\delta_{ij}$  is the Kronecker symbol. This implies

$$d\xi_i = \eta_i \sqrt{2\mathcal{D}\varepsilon_t}, \quad (16)$$

which, with the particularly meaningful choice  $\varepsilon_t = |dt|$ , gives

$$d\xi_i = \eta_i \sqrt{2\mathcal{D}|dt|}. \quad (17)$$

The two possibilities for the elementary displacements are therefore

$$dt > 0 \quad |dt| = dt, \quad d\xi_{i+} = \eta_i \sqrt{2\mathcal{D}dt}, \quad (18)$$

and

$$dt < 0 \quad |dt| = -dt, \quad d\xi_{i-} = \eta_i \sqrt{-2\mathcal{D}dt}, \quad (19)$$

which gives

$$\langle d\xi_{i\pm} d\xi_{j\pm} \rangle = \pm 2\delta_{ij} \mathcal{D}dt. \quad (20)$$

We have thus demonstrated how Eq. (20), from which the derivation of the Schrödinger equation proceeds [1, 8], can be obtained naturally with the above particularly meaningful choice for the scale variable.

Recall that a generalized total derivative

$$\hat{d}/dt = \partial/\partial t + \mathcal{V} \cdot \nabla - i\mathcal{D}\Delta \quad (21)$$

derives from Eq. (20), and that the fundamental equation of dynamics for the fluid of fractal paths (which are geodesics when there is no exterior field) can be written in terms of this “covariant” derivative

$$m \frac{\hat{d}\mathcal{V}}{dt} = -\nabla\phi. \quad (22)$$

Then, introducing the logarithm of the wave function  $\psi$  as a potential for the velocity field of paths,

$$\mathcal{V} = -2i\mathcal{D}\nabla \ln \psi, \quad (23)$$

the equation of dynamics can be subsequently integrated in terms of a Schrödinger equation:

$$\mathcal{D}^2 \Delta \psi + i\mathcal{D} \frac{\partial \psi}{\partial t} - \frac{\phi}{2m} \psi = 0. \quad (24)$$

In the quantum mechanical case,  $\mathcal{D}$  is related to the Compton length of the particle,  $\lambda_c$ , by  $\mathcal{D} = c\lambda_c/2 = \hbar/2m$  and Eq. (24) becomes the standard Schrödinger equation of quantum mechanics.

## 4 Conclusions

In this paper, we have attempted to make clearer the nature of the scale variables in the theory of Scale Relativity, and to analyze in more detail their effect on the fundamental laws of motion. A continuous but non-differentiable space (more generally space-time) is fractal, which means that its geometric description involves an explicit dependence on scale variables.

From the viewpoint of the theoretical description, the differential elements which one makes tending to zero are the fundamental scale variables. The

possibility to define these differentials is a direct consequence of the *continuity* of the space-time manifold. Therefore the non-differentiability manifests itself, not by making us unable to differentiate, but by the fact that ratios of differential elements, like  $dx/dt$ , no longer exist in the limit  $dt \rightarrow 0$ .

The Scale Relativity method solves this problem by simply taking into account all what happens when  $dt \rightarrow 0$ , but without taking the limit. In other words,  $v = dX/dt$  is considered as an explicit function  $v(dt)$  of  $dt$ . When the limit  $dt = 0$  does exist, this is the differentiable case, and it is included as a particular case of the new differential calculus. When it does not exist, the new method uses still a description tool in terms of the function  $v(dt)$ , and it simply means that  $v(dt) \rightarrow \infty$  when  $dt \rightarrow 0$ .

Another fundamental scale variable results from the fact that our access to physical phenomena is always made through measurements, and that a measurement apparatus is always characterized by a measurement resolution  $\varepsilon > 0$ . We have shown in the present work that the fundamental two-valuedness of velocities (time-derivatives) which gives rise to the complex nature of the wave function in the scale-relativistic foundation of quantum mechanics is fundamentally issued from the relation between these two scale variables,  $dt$  and  $\varepsilon_t$ .

We will, in a companion paper, generalize these results to space differentials and to the (motion-)relativistic case which lead to bi-spinorial (bi-quaternionic) wave function solutions of the Klein-Gordon and Dirac equations [10].

Moreover, it has been shown that the fundamental nature of scale variables is tensorial [8] and this fact plays a central role in the application of the scale relativity approach to gauge fields [12]. A generalization of the present results to this case will be made in a forthcoming work.

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